A DEFAULT SYSTEM WITH OVERSPILLING CONTAGION

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Abstract. In classical contagion models, default systems are Markovian conditionally on the observation of their stochastic environment, with interacting intensities. This necessitates that the environment evolves autonomously and is not influenced by the history of the default events. We extend the classical literature and allow a default system to have a contagious impact on its environment. In our framework, contagion can either be contained within the default system (i.e., direct contagion from a counterparty to another) or spill from the default system over its environment (indirect contagion).

This type of model is of interest whenever one wants to capture within a model possible impacts of the defaults of a class of debtors on the more global economy and vice versa.

1. Motivation and aims

Default events tend to cluster in time, a phenomenon that arises from diverse causes. The literature on dynamic modelling of defaults proposed so far two major mechanisms that produce this effect. First, there is the so-called cyclical correlation, i.e., the dependence of the debtors' financial situation on some common factors. One can naturally think of some macroeconomic factors that impact the default probabilities of many debtors at a time, as the interest rates, the prices of some commodities or the business cycle; a purely statistical approach using abstract or unobserved factors is also possible. Secondly, there is the so-called counterparty risk or direct contagion, i.e., the default of one debtor represents itself a destabilising factor impacting the default rates of surviving debtors (the counterparties of a defaulted debtor). In order to have these two mechanisms of contagion operational simultaneously it is necessary to distinguish within the model the default system from its environment. The role of the random environment is to carry the cyclical dependence. The most classical approach so far was to consider that conditionally on a given random environment, the vector of default indicator processes of the different debtors is a time inhomogeneous Markov chain.

While the Markovian assumption is convenient, it necessitates that the environment evolves autonomously and is not influenced by the history of the default events. Our aim here is precisely to remove this assumption. This equates to introducing a new source of contagion, that we call overspilling (or indirect) contagion: the one that transmits from the default system to its environment, subsequently having a feedback effect on the system itself. The construction, by its nature is not Markovian, the default probabilities depend not only on the current state of the default system, but also on the circumstances of the occurrence of the past defaults, more precisely the knowledge of their impact on the environment. The theory of enlargements of filtrations reveals to be the right tool to deal with this non Markovian framework.

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The paper is organised as follows. In Section 2 we introduce the precise Markovian setting that we aim to extend, and review the existing literature. Section 3 presents the model with overspilling contagion. Starting from a conditionally independent default system, the construction is obtained via a suitable change of the probability measure. We chose to develop our construction in a simplified framework. This has the advantage to shed light on the construction itself, while keeping the complexity of the model at a reasonable scale. Section 4 introduces and comments on the main result of the paper, that is, the survival probabilities for arbitrary sets of debtors can be obtained from a system of stochastic differential equations that can be solved recursively. Importantly, these equations are depending on the initial state of the system and the evolution of the environment, as in the Markovian setting. Section 5 is dedicated to the proof of the main result. For the reader's convenience, the classical results form the theory of enlargements of filtrations that were used in the proofs are listed in an appendix

2. Default models with interacting intensities: the Markovian approach

The mathematical description of the default contagion in a population of debtors has been inspired by the models of interacting particles systems studied in statistical mechanics. We here describe a model with n debtors, following Frey and Backhaus [17] (more related literature is found at the end of this section). Our aim is to introduce already the notations and framework that will be used in our extension, while reviewing the Markovian setup.

Let $(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual assumptions. The filtration $\mathbb{F}$ carries the relevant information about the environment of the default system. The default system itself is modelled by a multivariate process $Y = (Y_t(1), ..., Y_t(n))_{t\geq 0}$ with state space $I := \{0,1\}^n$, where 0 is the no default state is and 1 is the default state, so that $Y_t(k)$ is the indicator process of the default of the debtor k.

We denote $\mathcal{N} := \{1, ..., n\}$. The global information $\mathbb{G}^{\mathcal{N}} = (\mathcal{G}_t^{\mathcal{N}})_{t\geq 0}$ contains both the environment and the default system:

$$\mathcal{G}_t^{\mathcal{N}} := \mathcal{H}_{t^+}^{\mathcal{N}} \text{ with } \mathcal{H}_t^{\mathcal{N}} := \mathcal{F}_t \bigvee_{k\in\mathcal{N}} \sigma(Y_s(k), s \leq t). \tag{2.1}$$

The classical model is as follows: conditionally on $\mathcal{F}_\infty$, the default process Y is a time inhomogeneous Markov chain. The direct contagion mechanism is explicitly modelled as being driven by the process Y: the instantaneous transition rates of Y from state $x \in I$ to state $y \in I$ at time t are of the form:

$$q_t(x,y) = \begin{cases} f_t(x,y) \text{ if for some } k \in \mathcal{N} : y = x^k \text{ and } x(k) = 0 \\ 0 \text{ else,} \end{cases}$$

where $x^k \in I$ is obtained from $x \in I$ by flipping the k^{th} coordinate, $x(k)$. In other words, the transition rate is non zero only when y can be obtained from x by flipping a single element of x from 0 to 1. For any $k \in \mathcal{N}$ and $x \in I$ with $x(k) = 0$ the process $(f_t(x,x^k), t \geq 0)$ is considered $\mathbb{F}$ adapted and positive. It represents the default rate of the k^{th} debtor at a given time t, given that $Y_t = x$.

Remark. We observe that with the above specification, only one default can occur at a time and the default state is absorbing for all debtors. We shall keep these features in our extension.

One can define the default times of each debtor as:

$$\tau(k) := \inf\{t \geq 0 \mid Y_t(k) = 1\}, \quad k \in \mathcal{N}.$$

The default intensity of k^{th} obligor is a process $\lambda^{\mathcal{N}}(k)$ such that $\left(Y_t(k) - \int_0^{t\wedge\tau(k)} \lambda_s^{\mathcal{N}}(k)ds\right)$ is a martingale. The intensity of one debtor depends implicitly on the set of contagious debtors $\mathcal{N}$. Given the above transition rates, the intensities are a function of the default process:

$$\lambda_t^{\mathcal{N}}(k) = f_t(Y_t, Y_t^k).$$

One popular form of the intensities is the following:

$$\lambda_t^{\mathcal{N}}(k) = \lambda_t(k) + \sum_{i\in\mathcal{N}} \xi_t(k,i)Y_t(k), \tag{2.2}$$

where the processes $\lambda(k)$, $k \in \mathcal{N}$ and $\xi(k,i)$, $i,j \in \mathcal{N}$ are $\mathbb{F}$ adapted. As the default event is an absorbing state, we can rewrite (2.2) as:

$$\lambda_t^{\mathcal{N}}(k) = \lambda_t(k) + \sum_{i\in\mathcal{N}} \xi_t(k,i)\mathbf{1}_{\{\tau(k)\leq t\}}. \tag{2.3}$$

Default intensities as in (2.3) will arise as a special case in the construction that we propose in the next section.

Early default models with interacting intensities are Kusuoka [34], Davis and Lo [11], Jarrow and Yu [26] and Bielecki and Rutkowski [5]. The literature on default contagion grown exponentially in more recent years, we mention for instance Frey and Backhaus [17], [18], Herbertsson [23], Herbertsson and Rootzén [25], Jian and Zen [33], Bielecki, Crépey, Jeanblanc [3] to name only a few.

We point out that many existing models of default contagion are variants of the above described framework. Without being exhaustive, we mention some variants: non absorbing states (Giesecke and Weber [21], [22]); credit migration models with more than two states for each debtor (Davis and Esparragoza-Rodriguez [10], Bielecki et al. [4]); the so-called frailty models where the filtration $\mathbb{F}$ is (partially) unavailable for pricing and filtering techniques are used (Frey and Schmidt [20]); more than one default is allowed to occur at a time (Bielecki, Cousin, Crépey, Herbertsson [2]). We reccommand the survey paper by Bielecki, Crépey, Herbertsson [24] for a more detailed analysis of the Markovian setting.

Frey and Runggaldier [19] present a non Markovian default model, where, as in our approach, default events can occur simultaneously with somme events in the filtration $\mathbb{F}$. Their model involves unobserved factors and the focus is to propose the appropriate filtering techniques.

Finally, let us point out the paper of El Karoui et al. [15], which analyses the effects of changes of a probability measure for a default system. Their framework is very general and flexible to encompass many possible concrete applications: the default times do not necessarily admit an intensity, they can be either ordered or not ordered, finally it accommodates many possible information sets (i.e., observations of the default system). On the opposite, our objective in this paper is very applied: we propose a specific example of a default system that "contaminates" its environment which is a generalisation of the Markovian model presented above; we then characterise the corresponding survival probabilities.

3. Interacting intensities and overspilling contagion

As in the previous section, we consider a group $\mathcal{N} = \{1, ..., n\}$ of debtors. We shall introduce the dependence structure within the group $\mathcal{N}$ in two steps, as follows. To begin with, we built the model under a measure $\mathbb{P}^0$ where the default events are independent conditionally on $\mathbb{F}$, that is, we have cyclical correlation but no contagion. The channels for the transmission of the impacts from the default system on the environment are already present, but inactive under $\mathbb{P}^0$; they are materialised in a sequence of $\mathbb{F}$ stopping times $T(k)_{k\geq 0}$, where default events can occur with positive probability. We then shape the wished contagion (direct and indirect) via a change of the probability measure.

Consider a set $\mathcal{C} \subset \mathcal{N}$ of all debtors that are contagious, i.e., their default can produce a direct or an indirect contagion. The contagion mechanism that we propose is generating default intensities of the following form:

$$\lambda_t^{\mathcal{C}}(i) = \lambda_t(i) + \sum_{j \in \mathcal{C}} \xi_t^{X(j)}(i,j) \mathbf{1}_{\{\tau(j) < t\}} \quad \text{for } i \in \mathcal{N}, \tag{3.1}$$

where $X(j) \in \{A, B\}$ is a random variable, $X(j) = A$ if the default j is producing a direct contagion, while $X(j) = B$ will indicate that we have indirect contagion. The quantities $\xi_t^A(i,j)$ and $\xi_t^B(i,j)$ are a priori different quantities, but more importantly, when $X(j) = B$ some changes are occurring in the environment, i.e., some $\mathbb{F}$ adapted processes are impacted at the default event $\tau(j)$. The fact that the intensity of a surviving debtor i is augmented by $\xi^B(i,j)$ as shown in (3.1) is in fact a consequence of the modification of the environment. No impact on the environment occurs in the alternative case where $X(j) = A$.

We see that in such a framework, the environment does not evolve autonomously from the default system, which is precisely our objective.

Remark. In a Markovian model, the vector of intensity processes encodes the necessary and sufficient information about the distribution of the default process Y conditionally on $\mathbb{F}$ and given Y_0 (the $\mathbb{F}$ conditional transition rates can be obtained from $\lambda^{\mathcal{N}}$ and vice-versa). For this reason, these models are also called "intensity based". This is not the case in our framework, where we need to rely on the so-called hazard processes; a given intensity process can arise from different hazard processes. For this reason we do not provide immediately more details on the processes in (3.1), that we consider to be by-products of the model.

3.1. The model under $\mathbb{P}^0$: conditional independence.

We begin with a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^0)$ and $\mathbb{F}$ adapted and increasing process $\Gamma = (\Gamma(k), k \in \mathcal{N})$, with $\Gamma_0 = 0$ a.s. and $\lim_{t\to\infty} \Gamma_t = +\infty$ a.s.. We assume the probability space supports a sequence of random variables $e(k)$, $k \in \mathcal{N}$ which are i.i.d. with exponential distribution with parameter 1, and which are independent of $\mathcal{F}_\infty$. We define:

$$\tau(k) = \inf \{t \geq 0;\ \Gamma_t(k) \geq e(k)\},\ k \in \mathcal{N}.$$

The process Γ is known as the hazard process in the credit risk literature (see [16], [27], [9]); it synthesises all the necessary information about the default time; the compensator process of the default time can be computed starting from the hazard process, as we shall see in a moment.

For any set $\mathcal{C} \subset \mathcal{N}$, we introduce a filtration $\mathbb{G}^{\mathcal{C}}$ as

$$\mathcal{G}_t^{\mathcal{C}} := \mathcal{H}_{t+}^{\mathcal{C}} \text{ with } \mathcal{H}_t^{\mathcal{C}} := \mathcal{F}_t \bigvee_{k \in \mathcal{C}} \sigma(t \wedge \tau(k)),$$

i.e., the progressively enlarged filtration that satisfies the usual conditions and makes any $\tau(k)$ with $k \in \mathcal{C}$ a stopping time. We have $\mathbb{G}^{\emptyset} = \mathbb{F}$ and $\mathbb{G}^{\mathcal{N}}$ is as in (2.1).

Let us point out that in general, whenever single elements $\{i\}$ of $\mathcal{N}$ appear a superscripts, we shall omit the brackets. That is: we write $\mathbb{G}^i$ instead of $\mathbb{G}^{\{i\}}$, $\mathbb{G}^{\mathcal{C} \cup i}$ instead of $\mathbb{G}^{\mathcal{C} \cup \{i\}}$ etc.

For simplicity we shall assume that

$$\Gamma_t(k) = \int_0^t \alpha_s(k) ds + \Delta \Gamma_{T(k)}(k) \mathbf{1}_{\{T(k) \leq t\}},$$

where every $T(k), k \in \mathcal{N}$ is an $\mathbb{F}$ stopping time that is totally inaccessible, with intensity process $(\gamma_t(k))_{t \geq 0}$. The processes α and γ are nonegative and bounded. We define the $(\mathbb{F}, \mathbb{P}^0)$ martingales:

$$n_t(k) := \mathbf{1}_{\{T(k) \leq t\}} - \int_0^{t \wedge T(k)} \gamma_s(k) ds$$

These assumptions permit to have a simple model, where the default times admit an intensity. The more general framework appears in Coculescu [7], where only the case of a single debtor is treated. In the same spirit of having easy to handle formulas, we adopt the following additional assumptions:

Assumptions.

1. The martingales $n(k)$ and $n(j)$ are orthogonal for any $k, j \in \mathcal{N}$ with $k \neq j$.
2. For any $k \in \mathcal{N}$ the martingale:

$$p_t(k) := \mathbb{P}^0\Big(\tau(k) = T(k) | \mathcal{F}_t\Big), \text{ for } k \in \mathcal{N}$$

is orthogonal to $n(k)$.

Our construction under the measure $\mathbb{P}^0$ leads to $\mathbb{F}$ conditional survival probabilities (also known as the Azéma's supermartingales) that have simple forms, in particular for the k^{th} default:

$$Z_t(k) := \mathbb{P}(\tau(k) > t | \mathcal{F}_t) = e^{-\Gamma_t(k)}.$$

Below, we give the multiplicative decomposition of the survival probability, as well as the intensity of a default time $\tau(k), k \in \mathcal{N}$ corresponding to the above introduced hazard process:

Proposition 3.1. *The Azéma's supermartingale $Z(k)$ has a multiplicative decomposition (i.e., in the form of a local martingale times a decreasing and predictable process) given by:*

$$Z_t(k) = \mathcal{E}_t(\nu(k)) e^{-\Lambda_t(k)}, \tag{3.2}$$

where:

$$\begin{aligned} \nu_t(k) &:= -\int_0^t g_s(k) dn_s(k) \\ g_t(k) &:= p_t(k) e^{\Gamma_{t^-}(k)} \mathbf{1}_{\{T(k) \geq t\}} \\ \Lambda_t(k) &:= \int_0^t \lambda_s(k) ds \end{aligned}$$

with $\lambda(k)$, the intensity of $\tau(k)$, being:

$$\lambda_t(k) := \alpha_t(k) + g_t(k)\gamma_s(k).$$

Proof. We notice that $Z_t(k) = \mathbb{P}(\tau(k) > t|\mathcal{F}_t)$ gives:

$$\Delta Z_{T(k)}(k) = -\mathbb{P}^0(\tau(k) = T(k)|\mathcal{F}_{T_k}) = -p_{T(k)}(k).$$

On the other hand, $Z_t(k) = e^{-\Gamma_t(k)}$ gives:

$$\Delta Z_{T(k)}(k) = e^{-\Gamma_{T(k)}(k)} - e^{-\Gamma_{T(k)^-}(k)},$$

so that equalizing the two expressions, we obtain:

$$\Delta\Gamma_{T(k)} = -\ln\left(1 - p_{T(k)}(k)e^{\Gamma_{T(k)^-}(k)}\right) = -\ln\left(1 - g_{T(k)}(k)\right) = -\ln\left(1 + \Delta\nu_{T(k)}(k)\right)$$

Therefore, as the only discontinuity of $Z(k)$ occurs at $T(k)$, we have:

$$Z_t(k) = e^{-\int_0^t \alpha_s(k)ds} \prod_{s \le t} \left(1 + \Delta\nu_s(k)\right),$$

which is precisely (3.2). By assumption, the martingales $p(k)$ and $n(k)$ are orthogonal, and this ensures that $\nu(k)$ is a local martingale. The fact that the process $\lambda(k)$ is precisely the intensity of $\tau(k)$ follows from a result by Jeulin and Yor (1978) that we recall in the Appendix (Theorem A.2). $\square$

In order to be able later to discriminate between direct resp. indirect contagions, we need to decompose a default time $\tau(k)$ in its "specific" ($\tau^A(k)$) resp "systematic" ($\tau^B(k)$) counterparts, as follows:

Proposition 3.2. *Let us consider a set $\mathcal{C} \subset \mathcal{N}$ and fix some $k \in \mathcal{C}$. We define the $\mathbb{G}^{\mathcal{C}}$ stopping times $\tau^A(k)$ and $\tau^B(k)$:*

$$\begin{aligned}\tau^A(k) :&= \tau(k)\mathbf{1}_{\{\tau(k)\neq T(k)\}} + \infty\mathbf{1}_{\{\tau(k)=T(k)\}}\\ \tau^B(k) :&= \tau(k)\mathbf{1}_{\{\tau(k)=T(k)\}} + \infty\mathbf{1}_{\{\tau(k)\neq T(k)\}},\end{aligned}$$

so that:

$$\tau(k) = \tau^A(k) \wedge \tau^B(k).$$

Then, both τ^A and τ^B admit $(\mathbb{G}^{\mathcal{C}}, \mathbb{P}^0)$ intensities. These are given as follows.

(i) *For $\tau^A(k)$ the intensity is $\alpha(k)$.*

(ii) *for $\tau^B(k)$ the intensity is:*

$$\beta_t(k) := g_t(k)\gamma_t(k).$$

Proof. Using integration by parts in (3.2), we obtain:

$$\mathbb{P}^0(\tau(k) \le t|\mathcal{F}_t) = \int_0^t Z_{s^-}\alpha_s(k)ds + p_{T(k)}\mathbf{1}_{\{T(k)\le t\}}$$

On the other hand:

$$\begin{aligned}\mathbb{P}^0(\tau(k) \le t|\mathcal{F}_t) &= \mathbb{P}^0(\tau(k) \le t, \tau(k) = \tau^A(k)|\mathcal{F}_t) + \mathbb{P}^0(\tau(k) \le t, \tau(k) = \tau^B(k)|\mathcal{F}_t)\\ &= \mathbb{P}^0(\tau^A(k) \le t|\mathcal{F}_t) + \mathbb{P}^0(T(k) \le t, \tau(k) = \tau^B(k)|\mathcal{F}_t)\\ &= \mathbb{P}^0(\tau^A(k) \le t|\mathcal{F}_t) + \mathbf{1}_{\{T(k)\le t\}}p_t(k).\end{aligned}$$

From the two above expressions and using the property $p_t(k) = p_{t\wedge T(k)}$ (Corollary 5.2), we get:

$$\begin{aligned}\mathbb{P}^0(\tau^A(k) > t|\mathcal{F}_t) &= 1 - \int_0^t Z_{s^-}(k)\alpha_s(k)ds \\ \mathbb{P}^0(\tau^B(k) > t|\mathcal{F}_t) &= 1 - \int_0^t p_s(k)dn_s(k) - \int_0^t Z_{s^-}(k)g_s(k)\gamma_s(k)ds.\end{aligned} \tag{3.3}$$

The above expressions are the Doob Meyer decompositions of the Azéma supermartingales associated with $\tau^A(k)$ and $\tau^B(k)$. The corresponding intensities can be found by applying the formula of the compensator in Theorem A.2. □

Corollary 3.3. *Let $k \in \mathcal{N}$. The $\mathcal{G}^{\mathcal{N}}$-stopping time $\tau^A(k)$ avoids all finite stopping times of the filtration $\mathbb{G}^{\mathcal{N}-k}$, that is $\mathbb{P}^0(\tau^A(k) = \theta) = 0$ for any finite $\mathbb{G}^{\mathcal{N}-k}$-stopping time θ.*

Proof. We compute the Azéma supermartingale relative to $\tau^A(k)$ and the filtration $\mathbb{G}^{\mathcal{N}-k}$. Because $e(i), i \in \mathcal{N}$ are independent, and using (3.3) we have for $t \geq 0$, we obtain:

$$\begin{aligned}\mathbb{P}^0(\tau^A(k) > t|\mathcal{G}_t^{\mathcal{N}-k}) &= \mathbb{E}^0\left[\mathbb{P}^0\left(\tau^A(k) > t|\mathcal{F}_t \vee_{i\in\mathcal{N}-k} \sigma(e(i))\right)|\mathcal{G}_t^{\mathcal{N}-k}\right] \\ &= \mathbb{E}^0\left[\mathbb{P}^0\left(\tau^A(k) > t|\mathcal{F}_t\right)|\mathcal{G}_t^{\mathcal{N}-k}\right] \\ &= 1 - \int_0^t Z_{s^-}(k)\alpha_s(k)ds.\end{aligned}$$

This shows that the Azéma supermartingale relative to $\tau^A(k)$ and the filtration $\mathbb{G}^{\mathcal{C}-k}$ is continuous for finite t (in fact, one can show that there is a unique discontinuity at $t = \infty$, see [7] for more details). From Theorem VI.76. in [14] we deduce that $\tau^A(k)$ avoids any finite $\mathbb{G}^{\mathcal{N}-k}$-stopping time. □

Before proceeding to the next step and introduce contagion, it is useful to have a look to the survival probabilities under conditional independence, as seen from time 0. The aim is to emphasise that a class of probability measures is handy to use. Under $\mathbb{P}^0$, the time t survival probability in a group $\mathcal{C} \subset \mathcal{N}$ is given by:

$$\begin{aligned}\mathbb{P}^0(\tau(k) > t,\ \forall k \in \mathcal{C}) = \mathbb{E}^0\left[\prod_{k\in\mathcal{C}} Z_t(k)\right] &= \mathbb{E}^0\left[\exp\left(-\sum_{k\in\mathcal{C}}\int_0^t \lambda_s(k)ds\right)\prod_{k\in\mathcal{C}}\mathcal{E}_t(\nu(k))\right] \\ &= \bar{\mathbb{E}}_{\mathcal{C}}\left[\exp\left(-\sum_{k\in\mathcal{C}}\Lambda_t(k)\right)\right],\end{aligned} \tag{3.4}$$

with $\bar{\mathbb{E}}_{\mathcal{C}}$ being the expectation operator under the measure $\bar{\mathbb{P}}_{\mathcal{C}}$ defined below.

Definition 3.4. For $\mathcal{C} \subset \mathcal{N}$, we define a corresponding default adjusted probability measure, denoted by $\bar{\mathbb{P}}_{\mathcal{C}}$ and defined by:

$$\left.\frac{d\bar{\mathbb{P}}_{\mathcal{C}}}{d\mathbb{P}^0}\right|_{\mathcal{G}_t^{\mathcal{N}}} = \prod_{k\in\mathcal{C}}\mathcal{E}_t(\nu(k)), \quad t \geq 0.$$

The probability is well defined for all $\mathcal{C}$ as we have already assumed the processes α and γ to be bounded.

We summarise the $(\mathbb{G}^{\mathcal{C}}, \mathbb{P}^0)$ martingales that will play a role in the remaining:

$$m_t(k) := \mathbf{1}_{\{\tau^A(k) \leq t\}} - \int_0^{t\wedge\tau(k)} \alpha_s(k) ds, \quad t \geq 0 \tag{3.5}$$

$$n_t(k) = \mathbf{1}_{\{T(k) \leq t\}} - \int_0^{t\wedge\tau(k)} \gamma_s(k) ds, \quad t \geq 0. \tag{3.6}$$

3.2. **Contagion via a change of the probability measure.** In this subsection we propose filtered probability spaces

$$(\Omega, \mathcal{G}, \mathbb{G}^{\mathcal{N}}, \mathbb{P}^{\mathcal{C}}), \mathcal{C} \subset \mathcal{N}$$

that represent models for different sets $\mathcal{C}$ of contagious debtors.

We first introduce the impact matrices: $(\phi_t^A(i,j))_{(i,j)\in\mathcal{N}^2}$ and $(\phi_t^A(i,j))_{(i,j)\in\mathcal{N}^2}$, with components being nonnegative and bounded processes that are $\mathbb{F}$-predictable. Here $\phi^A(i,j)$ (resp. $\phi^B(i,j)$) is the impact directly (resp. indirectly) induced by the default of the j^{th} debtor on the i^{th} debtor, whenever the last is not yet defaulted,

From now on, we assume that for all $i \in \mathcal{N}$, $p(i) \in [0,1)$. The following proposition is a simple application of the Girsanov's theorem.

Proposition 3.5. *Let $\mathcal{C}$ be the set of contagious debtors, $\mathcal{C} \subset \mathcal{N}$. We introduce for all $i \in \mathcal{C}$ the predictable processes:*

$$A_t^{\mathcal{C}}(i) := \frac{1}{\alpha_t(i)} \sum_{j\in\mathcal{C}} \phi_t^A(i,j) \mathbf{1}_{\{\tau^A(j) < t\}}, \quad t \geq 0 \tag{3.7}$$

$$B_t^{\mathcal{C}}(i) := \frac{1}{\gamma_t(i)} \sum_{j\in\mathcal{C}} \phi_t^B(i,j) \mathbf{1}_{\{\tau^B(j) < t\}}, \quad t \geq 0 \tag{3.8}$$

whenever $\alpha_t(i) > 0$ resp. $\gamma_t(i) > 0$; and consider $A_t^{\mathcal{C}}(i) = 0$ resp. $B_t^{\mathcal{C}}(i) = 0$ otherwise.

We define the family of probability measures $(\mathbb{P}^{\mathcal{C}}), \mathcal{C} \subset \mathcal{N}$:

$$\left.\frac{d\mathbb{P}^{\mathcal{C}}}{d\mathbb{P}^0}\right|_{\mathcal{G}_t^{\mathcal{N}}} = D_t^{\mathcal{C}} := \prod_{i\in\mathcal{N}} \mathcal{E}_t \left(\int_0^t A_s^{\mathcal{C}}(i) dm_s(i) \right) \prod_{i\in\mathcal{N}} \mathcal{E}_t \left(\int_0^t B_s^{\mathcal{C}}(i) dn_s(i) \right).$$

Then the default time $\tau(i)$, $i \in \mathcal{N}$ has the $(\mathbb{G}^{\mathcal{N}}, \mathbb{P}^{\mathcal{C}})$ intensity given by:

$$\lambda_t^{\mathcal{C}}(i) = \lambda_t(i) + \left\{ \alpha_t(i) A_t^{\mathcal{C}}(i) + \beta_t(i) B_t^{\mathcal{C}}(i) \right\}.$$

Remark. We notice that the default intensities under $\mathbb{P}^{\mathcal{C}}$ are of the form announced in (3.1):

$$\lambda_t^{\mathcal{C}}(i) = \lambda_t(i) + \sum_{j\in\mathcal{C}} \xi_t^{X(j)}(i,j) \mathbf{1}_{\{\tau(j) < t\}} \quad \text{for } i \in \mathcal{N},$$

with $X(j) = A\mathbf{1}_{\{\tau(j)=\tau^A(j)\}} + B\mathbf{1}_{\{\tau(j)=\tau^B(j)\}}$, which is a $\mathcal{G}_{\tau(j)}^{\mathcal{N}}$ measurable random variable; and $\xi^A(i,j) = \phi^A(i,j)$ and $\xi^B(i,j) = g(i)\phi^B(i,j)$.

4. Main result

We work under $(\Omega, \mathcal{G}, \mathbb{G}^{\mathcal{N}}, \mathbb{P}^{\mathcal{N}})$, that is, the class of contagious debtors is $\mathcal{N}$. This is without loss of generality: one can set the k^{th} column of the two impact matrices ϕ^A and ϕ^B to be null and render the k^{th} debtor non contagious.

We want to characterise the time t survival probabilities:

$$\mathbb{P}^{\mathcal{N}}(\tau(k) > t, \forall k \in \mathcal{C}) \text{ for any } \mathcal{C} \in \mathcal{N}.$$

We recall that under conditional independence, the survival probabilities satisfy:

$$\mathbb{P}^0(\tau(k) > t,\ \forall k \in \mathcal{C}) = \bar{\mathbb{E}}_{\mathcal{C}}\left[\ell_t\right],$$

where ℓ satisfies: $d\ell_t = -\ell_t(\sum_{k\in\mathcal{C}} \lambda_t(k))dt$ (see the expression in (3.4)). Our aim is to propose formulas under $\mathbb{P}^{\mathcal{N}}$ that have a similar form, that is:

$$\mathbb{P}^{\mathcal{N}}(\tau(k) > t,\ \forall k \in \mathcal{C}) = \bar{\mathbb{E}}_{\mathcal{C}}\left[\ell_t\right], \tag{4.1}$$

where ℓ is an $\mathbb{F}$ adapted process. But now, ℓ belongs to a larger family of processes that arises as solution of a system of linear stochastic differential equations that can be solved recursively. This is the object of Theorem 4.1 below, which is the main result of this paper.

It would be tempting to denote the process ℓ appearing in (4.1) $\ell^{\mathcal{C}}$, to reflect that it corresponds to the survival probabilities in the group $\mathcal{C}$. However, we refrain from doing so; instead our notation will be: $\ell = \ell^{\mathcal{N}-\mathcal{C}}$. We make the choice that subsets of $\mathcal{N}$ appearing as superscripts indicate the contagious entities. Indeed, we observe that:

$$\begin{aligned}
\mathbb{P}^{\mathcal{N}}(\tau(k) > t,\ \forall k \in \mathcal{C}) &= \mathbb{P}^{\mathcal{N}-\mathcal{C}}(\tau(k) > t,\ \forall k \in \mathcal{C}) && (4.2)\\
&= \bar{\mathbb{E}}_{\mathcal{C}}\left[\ell_t^{\mathcal{N}-\mathcal{C}}\right], && (4.3)
\end{aligned}$$

i.e., we can consider that $\mathcal{N}-\mathcal{C}$ is in fact the set of contagious debtors when computing the above probability. This is because under $\mathbb{P}^{\mathcal{N}}$, the contagion produced by a particular debtor occurs only after its default and is inexistent before. In mathematical terms:

$$D_t^{\mathcal{N}} \mathbf{1}_{\{\tau(k)>t,\ \forall k\in\mathcal{C}\}} = D_t^{\mathcal{N}-\mathcal{C}} \mathbf{1}_{\{\tau(k)>t,\ \forall k\in\mathcal{C}\}}.$$

as appearing from the expressions in Proposition 3.5.

Additional notation. Given a vector $(V(i), i \in \mathcal{N})$ and a matrix $M = (M(i,j), i, j \in \mathcal{N})$ and with $\mathcal{C}, \mathcal{D} \subset \mathcal{N}$ we write

$$V(\mathcal{C}) := \sum_{i\in\mathcal{C}} V(i) \quad M(\mathcal{C}, \mathcal{D}) := \sum_{i\in\mathcal{C}} \sum_{j\in\mathcal{D}} M(i,j).$$

For instance $\lambda_t(\mathcal{C}) = \sum_{i\in\mathcal{C}} \lambda_t(i)$ and $\phi_t^A(\mathcal{C}, \mathcal{D}) = \sum_{i\in\mathcal{C}} \sum_{j\in\mathcal{D}} \phi_t^A(i,j)$, etc.

Theorem 4.1. *Suppose that $\mathcal{C}, \mathcal{D} \in \mathcal{N}$ with $\mathcal{C} \cap \mathcal{D} = \emptyset$ and denote $\mathcal{S} := \mathcal{N} - \mathcal{C}$. Then:*

$$\mathbb{P}^{\mathcal{N}}(\tau(k) > t, \forall i \in \mathcal{C}\ ;\ \tau^B(j) \le t, \forall j \in \mathcal{D}) = \bar{\mathbb{E}}_{\mathcal{C}}[\ell_t^{\mathcal{C}|\mathcal{D}}],$$

where $\ell^{\mathcal{S}|\mathcal{D}}$ *satisfies:*

$$d\ell_t^{\mathcal{S}|\mathcal{D}} = \left\{ -\ell_{t^-}^{\mathcal{S}|\mathcal{D}} \lambda_t(\mathcal{C}) + \sum_{j \in \mathcal{S}-\mathcal{D}} \left(\ell_{t^-}^{\mathcal{S}|\mathcal{D}} - \ell_{t^-}^{\mathcal{S}-j|\mathcal{D}} + \ell_{t^-}^{\mathcal{S}|\mathcal{D}\cup j} p_t(j) \mathbf{1}_{\{T(j)<t\}} \right) \psi^A(\mathcal{C} \cup \mathcal{D}, j) \right\} dt$$

$$\sum_{k \in \mathcal{N}} \left\{ \sum_{j \in \mathcal{S}} \mathbf{1}_{\{T(j)<t\}} \left(\mathbf{1}_{\{j \in \mathcal{D}\}} \ell_{t^-}^{\mathcal{S}|\mathcal{D}} + \mathbf{1}_{\{j \in \mathcal{S}-\mathcal{D}\}} \ell_{t^-}^{\mathcal{S}|\mathcal{D}\cup j} p_t(j) \right) \frac{\phi_t^B(k,j)}{\gamma_t(k)} \right\} dn_t(k) \qquad (4.4)$$

$$\ell_0^{\mathcal{S}|\mathcal{D}} = 1.$$

Above, we have denoted:

$$\psi^A(k,j) := \begin{cases} \phi^A(k,j) & k \in \mathcal{S} - \mathcal{D} \\ \phi^A(k,j) \mathbf{1}_{\{T(k)>t\}} & k \in \mathcal{D}. \end{cases}$$

In particular, denoting $\ell^{\mathcal{S}} := \ell^{\mathcal{S}|\emptyset}$, *the survival probability in group* $\mathcal{C}$ *satisfies:*

$$\mathbb{P}^{\mathcal{N}}(\tau(k) > t, \ \forall k \in \mathcal{C}) = \bar{\mathbb{E}}_{\mathcal{C}} \left[\ell_t^{\mathcal{S}} \right],$$

with:

$$d\ell_t^{\mathcal{S}} = \left\{ -\ell_{t^-}^{\mathcal{S}} \lambda_t(\mathcal{C}) + \sum_{j \in \mathcal{S}} \left(\ell_{t^-}^{\mathcal{S}} - \ell_{t^-}^{\mathcal{S}-j} + \ell_{t^-}^{\mathcal{S}|j} p_t(j) \mathbf{1}_{\{T(j)<t\}} \right) \phi^A(\mathcal{C}, j) \right\} dt$$

$$\sum_{k \in \mathcal{N}} \left\{ \sum_{j \in \mathcal{S}} \mathbf{1}_{\{T(j)<t\}} \ell_{t^-}^{\mathcal{S}|j} p_t(j) \frac{\phi_t^B(k,j)}{\gamma_t(k)} \right\} dn_t(k) \qquad (4.5)$$

$$\ell_0^{\mathcal{S}} = 1.$$

We postpone to the next section the proof of this result. For now, we want to explore the SDEs above.

We begin by emphasising some particular cases:

(1) If $\phi^A \equiv 0$ and $\phi^B \equiv 0$ (i.e., there is no contagion), then $\mathbb{P}^{\mathcal{C}} = \mathbb{P}^0$ and:

$$d\ell_t^{\emptyset} = -\ell_t^{\emptyset} \lambda_t(\mathcal{C}) dt$$

which corresponds indeed to the expression in (3.4).

(2) If $p(i) \equiv 0$ for all $i \in \mathcal{N}$ (i.e., there is no impact of the default system on its environment), then all $\tau(i), i \in \mathcal{N}$ avoid the $\mathbb{F}$ stopping times. We recover in this way a Markovian framework similar to the one introduced in Section 2, where the transition rate at time t from state $x \in \{0,1\}^n$ to state $y \in \{0,1\}^n$ is:

$$q_t(x,y) = \begin{cases} \lambda_t(k) + \sum_{j \in \mathcal{N}} \phi_t^A(k,j) x(j) \text{ if for some } k \in \mathcal{N} : y = x^k \text{ and } x(k) = 0 \\ 0 \text{ else,} \end{cases}$$

where, as in the previous section, $x^k \in \{0,1\}^n$ is obtained from $x \in I$ by flipping the k^{th} coordinate, $x(k)$.

We observe that $\bar{\mathbb{P}}_{\mathcal{C}} = \mathbb{P}^0$ and (4.5) becomes:

$$d\ell_t^{\mathcal{S}} = -\ell_t^{\mathcal{S}} \left\{ \lambda_t(\mathcal{C}) + \phi_t^A(\mathcal{C}, \mathcal{S}) \right\} dt + \sum_{j \in \mathcal{S}} \ell_t^{\mathcal{S}-j} \phi_t^A(\mathcal{C}, j) dt. \qquad (4.6)$$

(3) If $\phi^A = 0$ and $\phi^B \neq 0$ (i.e., there is only indirect contagion), then:

$$d\ell_t^{\mathcal{S}} = - \ell_{t^-}^{\mathcal{S}} \lambda_t(\mathcal{C})dt + \sum_{j \in \mathcal{S}} \mathbf{1}_{\{T(j)<t\}} \ell_{t^-}^{S|j} p_t(j) \sum_{k \in \mathcal{N}} \left(\frac{\phi_t^B(k,j)}{\gamma_t(k)} \right) dn_t(k).$$

We now indicate how one can concretely obtain the survival probabilities from the SDEs in Theorem 4.1. A target set $\mathcal{C}^* \subset \mathcal{N}$ is fixed, i.e., we want to obtain the process $\ell^{\mathcal{S}^*}$, with $\mathcal{S}^* = \mathcal{N} - \mathcal{C}^*$. We proceed by iteration, starting with $\mathcal{S} = \emptyset$ we recursively add elements so to create all possible subsets of $\mathcal{S}^*$. The set $\mathcal{S}^* = \mathcal{N} - \mathcal{C}^*$ is obtained at the last iteration. More precisely, this works as follows:

0. $\mathcal{S} = \emptyset$. We compute $\ell_t^{\emptyset}$.
1. For all $j \in \mathcal{S}^*$, we take $\mathcal{S} = \{j\}$ and obtain the quantities $\ell^{j|j}$ and ℓ^j.
2. For all $\{j_1, j_2\} \subset \mathcal{S}^*$, we take $\mathcal{S} = \{j_1, j_2\}$ and obtain the quantities $\ell^{S|S}, \ell^{S|j_1}, \ell^{S|j_2}, \ell^{\mathcal{S}}$ (in that order).

...

In general, at the k^{th} iteration:

k. For any $\mathcal{S} \subset \mathcal{S}^*$ with $\mathbf{card}(\mathcal{S}) = k$ and for any $\mathcal{D} \subset \mathcal{S}$, we compute $\ell^{\mathcal{S}|\mathcal{D}}$, in the decreasing order of the cardinality of $\mathcal{D}$. There are $\binom{n}{k}$ subsets of $\mathcal{S}^*$ that contain k elements, each of them having 2^k different subsets. Hence, at the k^{th} iteration, we have to solve $\binom{n}{k}2^k$ equations of the type (4.4). For solving these equations, the quantities obtained at step $k-1$ are needed.

For instance, if $\mathbf{card}(\mathcal{S}^*) = s$, the procedure necessitates iterations $0, 1, \cdots, s$ of the form described above, that is, we need to solve for:

$$\sum_{k=0}^{s} \binom{s}{k} 2^k = 3^s$$

equations of the type (4.4). We see that the complexity of the procedure is very high, when applied to default systems of big size.

In practical applications however, we advocate that the complexity can be reasonably reduced. In most financial systems, even though there are a multitude of debtors, the number of those defaults that are expected to have a notable impact outside the system itself is presumably limited to a few entities (the systemic firms). The other firms we can be considered as non systemic: we can assume that $\tau^B(k) = \infty$ $a.s.$ when debtor k is non systemic. The interpretation is that if debtor k is not systemic, its default has at most a direct contagious impact on its counterparties (i.e., the other debtors in the default system), but not a larger economic impact (i.e. on the environment of the default system).

For example, suppose that $\mathcal{S}^* = \mathcal{S}_A^* \cup \mathcal{S}_B^* \subset \mathcal{N}$ and for all $k \in \mathcal{S}_A^*$ we have $\tau^B(k) = \infty$ $a.s.$, that is $\tau(k) = \tau^A(k)$ $a.s.$. In other words, $\mathcal{S}_A^*$ is a group of non systemic debtors and $\mathcal{S}_B^*$ contains possibly systemic debtors. We consider $\mathcal{S}_A^* \cap \mathcal{S}_B^* = \emptyset$ and $\mathbf{card}(\mathcal{S}_B^*) = b$, so that $\mathbf{card}(\mathcal{S}_A^*) = s - b$. In order to obtain the process $\ell^{\mathcal{S}^*}$, we need this time to solve for:

$$\sum_{k=0}^{s-b} \binom{s-b}{k} \times \left(\sum_{k=0}^{b} \binom{b}{k} 2^k \right) = 2^{s-b} 3^b$$

equations of the type (4.4). The complexity of the procedure is considerably reduced when b small.

5. Proof of the main result

This section is dedicated to the proof of the Theorem 4.1. For the convenience of the reader, we gather separately, in Appendix A the basic results from the theory of the enlargement of filtrations that were useful for our proofs. Also for the sake of clarity, we establish some intermediary results in the first two subsections.

5.1. **Preparatory results (I).** Because we are dealing with numerous filtrations and probabilities, we clarify here what a martingale becomes when we change the filtration and/or probability. Only the relevant changes of filtration and probability are emphasised.

The results in Theorem 4.1 make appear expectations under $\bar{\mathbb{P}}_{\mathcal{C}}$. Let us fix a set $\mathcal{C} \in \mathcal{N}$. Under $\bar{\mathbb{P}}_{\mathcal{C}}$ we have that:

- For $k \in \mathcal{C}$, the stopping time $T(k)$ has an intensity $\gamma(k)(1-g(k))$. We define the following $(\mathbb{F}, \bar{\mathbb{P}}_{\mathcal{C}})$-martingales:

$$\bar{n}_t^{\mathcal{C}}(k) := \mathbf{1}_{\{T(k) \le t\}} - \int_0^{t \wedge T(k)} \gamma_s(k)(1-g_s(k))ds \tag{5.1}$$

- For $i \in \mathcal{N} - \mathcal{C}$, the stopping time $T(i)$ has unchanged intensity $\gamma(i)$, as under $\mathbb{P}^0$.
- More generally, all the $\mathbb{P}^0$-martingales orthogonal to $n(k), k \in \mathcal{C}$ are also $\bar{\mathbb{P}}_{\mathcal{C}}$ martingales.

Notation. Given two filtrations $\mathbb{F} \subset \mathbb{G}$ and a probability measure $\mathbb{P}$, we write $\mathbb{F} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$ when all $\mathbb{F}$ martingales remain $\mathbb{G}$ martingales under the probability measure $\mathbb{P}$. This property is usually called immersion property (i.e., we say that $\mathbb{F}$ is immersed in $\mathbb{G}$) or (H) hypothesis.

Lemma 5.1. *The following hold:*

(a) *For all $\mathcal{C} \subset \mathcal{N}$ we have the following property:*

$$\mathbb{F} \overset{\mathbb{P}^0}{\hookrightarrow} \mathbb{G}^{\mathcal{C}} \overset{\mathbb{P}^0}{\hookrightarrow} \mathbb{G}^{\mathcal{N}},$$

(b) *Let $(\mathcal{C}_k)$ be an increasing family of subsets of $\mathcal{N}$. Under the measure $\mathbb{P}^{\mathcal{C}_k}$ the following hold:*

$$\mathbb{F} \overset{\mathbb{P}^{\mathcal{C}_k}}{\not\hookrightarrow} \mathbb{G}^{\mathcal{C}_1} \overset{\mathbb{P}^{\mathcal{C}_k}}{\not\hookrightarrow} \dots \overset{\mathbb{P}^{\mathcal{C}_k}}{\not\hookrightarrow} \mathbb{G}^{\mathcal{C}_k} \overset{\mathbb{P}^{\mathcal{C}_k}}{\hookrightarrow} \dots \overset{\mathbb{P}^{\mathcal{C}_k}}{\hookrightarrow} \mathbb{G}^{\mathcal{N}},$$

(c) *Under $\bar{\mathbb{P}}_{\mathcal{C}}$ we have:*

$$\mathbb{F} \overset{\bar{\mathbb{P}}_{\mathcal{C}}}{\hookrightarrow} \mathbb{G}^{\mathcal{S}} \overset{\bar{\mathbb{P}}_{\mathcal{C}}}{\hookrightarrow} \mathbb{G}^{\mathcal{N}}, \ \textit{for any } \mathcal{S} \subset \mathcal{N}.$$

Proof. (a) Let us consider $X \in \mathcal{G}_\infty^{\mathcal{C}}$. We denote $\mathcal{H}_t := \mathcal{G}_t^{\mathcal{C}} \vee_{k \in \mathcal{N}-\mathcal{C}} \sigma(e(k))$. We have that $\mathcal{G}_t^{\mathcal{N}} \subset \mathcal{H}_t$ and because any $e(k), k \in \mathcal{N} - \mathcal{C}$ is independent from $\mathcal{G}_\infty^{\mathcal{C}}$, we obtain:

$$\mathbb{E}^0[X|\mathcal{G}_t^{\mathcal{N}}] = \mathbb{E}^0[\mathbb{E}^0[X|\mathcal{H}_t]\mathcal{G}_t^{\mathcal{N}}] = \mathbb{E}^0[\mathbb{E}^0[X|\mathcal{G}_t^{\mathcal{C}}]\mathcal{G}_t^{\mathcal{N}}] = \mathbb{E}^0[X|\mathcal{G}_t^{\mathcal{C}}].$$

To conclude, we apply Theorem A.4 (3).

(b) Under $\mathbb{P}^{\mathcal{C}}$ the $\mathbb{G}^{\mathcal{C}}$-compensators of the $\mathbb{F}$-stopping times $T(k), k \in \mathcal{N}$ are not adapted to sub-filtrations of $\mathbb{G}^{\mathcal{C}}$. This shows that:

$$\mathbb{F} \overset{\mathbb{P}^{\mathcal{C}_k}}{\not\hookrightarrow} \mathbb{G}^{\mathcal{C}_1} \overset{\mathbb{P}^{\mathcal{C}_k}}{\not\hookrightarrow} \dots \overset{\mathbb{P}^{\mathcal{C}_k}}{\not\hookrightarrow} \mathbb{G}^{\mathcal{C}_k}.$$

On the other hand, the Radon-Nikodým density process $\frac{d\mathbb{P}^{\mathcal{C}_k}}{d\mathbb{P}^0}|_{\mathcal{G}_t^{\mathcal{N}}}$ is $\mathbb{G}^{\mathcal{C}_k}$ adapted. Then, by Proposition A.5, the immersion property holds in the given superfiltrations of $\mathbb{G}^{\mathcal{C}_k}$, as it was holding under $\mathbb{P}^0$.

(c) The Radon-Nikodým density process $\frac{d\bar{\mathbb{P}}_{\mathcal{C}}}{d\mathbb{P}^0}|_{\mathcal{G}_t^{\mathcal{N}}}$ is $\mathbb{F}$ adapted. The result is then a consequence of Proposition A.5.

□

Corollary 5.2. *The* $(\mathbb{F}, \mathbb{P}^0)$ *martingales* $p(i)$ *satisfy* $p_t(i) = p_{t\wedge T(i)}(i)$.

Proof. For any $i \in \mathcal{N}$, $p(i)$ is the $(\mathbb{F}, \mathbb{P}^0)$ optional projection of the martingales $g(i) := \mathbb{P}^0(\tau(i) = T(i)|\mathcal{G}^i_\cdot)$ that satisfy $g_t(i) = g_{t\wedge T(i)} = \int_0^t \mathbf{1}_{\{T(i)>s\}} dg_s(i)$. From Lemma 5.1 (a), we have $\mathbb{F} \overset{\mathbb{P}^0}{\hookrightarrow} \mathbb{G}^i$ so that we can apply Theorem A.6 to conclude. □

5.2. **Preparatory results (II).** In this section, a set $\mathcal{C} \subset \mathcal{N}$ is fixed and we consider two additional sets:

$$\mathcal{S} := \mathcal{N} - \mathcal{C} \text{ and } \mathcal{D} \subset \mathcal{S}.$$

In Theorem 4.1, the SDE (4.4) for $\ell^{\mathcal{S}|\mathcal{D}}$ is obtained after projecting on the filtration $\mathbb{F}$ an $\mathbb{G}^{\mathcal{S}-\mathcal{D}}$ adapted process, denoted $L^{\mathcal{S}|\mathcal{D}}$. In this section, we identify the process $L^{\mathcal{S}|\mathcal{D}}$ and prepare the building blocks for obtaining its $(\mathbb{F}, \bar{\mathbb{P}}_{\mathcal{C}})$ projection.

Proposition 5.3. *The following hold:*

(a)

$$\mathbb{P}^{\mathcal{N}}(\tau(k) > t,\ \forall k \in \mathcal{C}) = \bar{\mathbb{E}}_{\mathcal{C}}[\ell_t^{\mathcal{S}}]. \tag{5.2}$$

where $\ell^{\mathcal{S}}$ *is the* $(\mathbb{F}, \bar{\mathbb{P}}_{\mathcal{C}})$ *optional projection of the process* $L^{\mathcal{S}}$ *defined by:*

$$L_t^{\mathcal{S}} := \exp\left(-\Lambda_t^{\mathcal{S}}(\mathcal{C})\right) \prod_{i\in\mathcal{S}} \mathcal{E}_t\left(\int_0^t A_s^{\mathcal{S}}(i) dm_s(i)\right) \prod_{i\in\mathcal{N}} \mathcal{E}_t\left(\int_0^t B_s^{\mathcal{S}}(i) d\bar{n}_s^{\mathcal{C}}(i)\right).$$

(b)

$$\mathbb{P}^{\mathcal{N}}(\tau(k) > t, \forall k \in \mathcal{C}\ ;\ \tau^B(j) \le t, \forall j \in \mathcal{D}) = \bar{\mathbb{E}}_{\mathcal{C}}\left[\ell_t^{\mathcal{S}|\mathcal{D}} \prod_{j\in\mathcal{D}} p_t(j) \mathbf{1}_{\{T(j)\le t\}}\right], \tag{5.3}$$

where $\ell^{\mathcal{S}|\mathcal{D}}$ *the* $(\mathbb{F}, \bar{\mathbb{P}}_{\mathcal{C}})$ *optional projection of the process* $L^{\mathcal{S}|\mathcal{D}}$ *defined by:*

$$L_t^{\mathcal{S}|\mathcal{D}} := \prod_{i\in\mathcal{S}-\mathcal{D}} \mathcal{E}_t\left(\int_0^\cdot A_s^{\mathcal{S}-\mathcal{D}}(i) dm_s(i)\right) \prod_{i\in\mathcal{N}} \mathcal{E}_t\left(\int_0^\cdot B_s^{\mathcal{S}|\mathcal{D}}(i) d\bar{n}_s^{\mathcal{C}}(i)\right) \tag{5.4}$$

$$\times \exp - \left(\Lambda_t^{\mathcal{S}|\mathcal{D}}(\mathcal{C}) + \sum_{j\in\mathcal{D}} \int_0^{T(j)\wedge t} A_s^{\mathcal{S}-\mathcal{D}}(j)\alpha(j) ds\right),$$

with:

$$\Lambda_t^{\mathcal{S}|\mathcal{D}}(i) := \int_0^t \lambda_s^{\mathcal{S}|\mathcal{D}}(i) ds \tag{5.5}$$

$$\lambda_t^{\mathcal{S}|\mathcal{D}}(i) := \lambda_t^{\mathcal{S}-\mathcal{D}}(i) + g_t(i) \left(\sum_{j\in\mathcal{D}} \phi_s^B(i,j) \mathbf{1}_{\{T(j)<s\}}\right) \tag{5.6}$$

$$B_t^{\mathcal{S}|\mathcal{D}}(i) := \frac{1}{\gamma(i)} \left(\sum_{j\in\mathcal{S}-\mathcal{D}} \phi_t^B(i,j) \mathbf{1}_{\{\tau^B(j)<t\}} + \sum_{j\in\mathcal{D}} \phi_t^B(i,j) \mathbf{1}_{\{T(j)<t\}}\right) \tag{5.7}$$

Proof. Let us denote: $p^{\mathcal{S}|\mathcal{D}}(t) := \mathbb{P}^{\mathcal{N}}(\tau(k) > t, \forall k \in \mathcal{C} \ ; \ \tau^B(j) \leq t, \forall j \in \mathcal{D})$. We first show that:

$$p^{\mathcal{S}|\mathcal{D}}(t) = \bar{\mathbb{E}}_{\mathcal{C}}\left[L_t^{\mathcal{S}}\mathbf{1}_{\{\tau^B(j)\leq t; \forall j\in\mathcal{D}\}}\right]. \tag{5.8}$$

Looking to the formula in (5.8), the roadmap is clear: we need to go form the probability $\mathbb{P}^{\mathcal{N}}$ to $\bar{\mathbb{P}}_{\mathcal{C}}$, and from the filtration $\mathbb{G}^{\mathcal{N}}$ to $\mathbb{G}^{\mathcal{S}}$, as $L^{\mathcal{S}|\mathcal{D}}$ is a $\mathbb{G}^{\mathcal{S}}$ adapted process. We notice that the Radon-Nikodým density process $D^{\mathcal{S}} = d\mathbb{P}^{\mathcal{S}}/d\mathbb{P}^0|_{\mathcal{G}^{\mathcal{N}}}$, with $\mathcal{S} \subset \mathcal{N}$ is $\mathbb{G}^{\mathcal{N}}$ adapted.

We now introduce some useful $\mathbb{G}^{\mathcal{S}}$-adapted processes:

$$\begin{aligned}
E_t^{\mathcal{S}} &= \prod_{i\in\mathcal{S}}\mathcal{E}_t\left(\int_0^t A_s^{\mathcal{S}}(i)dm_s(i)\right)\prod_{i\in\mathcal{N}}\mathcal{E}_t\left(\int_0^t B_s^{\mathcal{S}}(i)dn_s(i)\right)\\
F_t^{\mathcal{S}} &= \prod_{i\in\mathcal{S}}\mathcal{E}_t\left(\int_0^t A_s^{\mathcal{S}}(i)dm_s(i)\right)\prod_{i\in\mathcal{N}}\mathcal{E}_t\left(\int_0^t B_s^{\mathcal{S}}(i)d\bar{n}_s^{\mathcal{C}}(i)\right)\\
L_t^{\mathcal{S}} &= F_t^{\mathcal{S}}\exp\left(-\Lambda_t^{\mathcal{S}}(\mathcal{C})\right)
\end{aligned}$$

so that

$$L_t^{\mathcal{S}} = F_t^{\mathcal{S}}\exp\left(-\Lambda_t^{\mathcal{S}}(\mathcal{C})\right)$$

We remark that, indeed, both $E^{\mathcal{S}}$ and $F^{\mathcal{S}}$ are $\mathbb{G}^{\mathcal{S}}$ adapted. Also, $E^{\mathcal{S}}$ is a local martingale under $\mathbb{P}^0$, while $F^{\mathcal{S}}$ is a local martingale under $\bar{\mathbb{P}}_{\mathcal{C}}$. Furthermore, we have the relation:

$$F_t^{\mathcal{S}} = E_t^{\mathcal{S}} \times \exp\left(\sum_{k\in\mathcal{C}}\int_0^t B^{\mathcal{S}}(k)\beta_s(k)ds\right).$$

It follows that the expression in (b) can be computed as:

$$\begin{aligned}
p^{\mathcal{S}|\mathcal{D}}(t) &= \mathbb{E}^0\left[D_t^{\mathcal{N}}\mathbf{1}_{\{\tau(k)>t,\forall k\in\mathcal{C}\}}\mathbf{1}_{\{\tau^B(j)\leq t,\forall j\in\mathcal{D}\}}\right]\\
&= \mathbb{E}^0\left[D_t^{\mathcal{S}}\mathbf{1}_{\{\tau(k)>t,\forall k\in\mathcal{C}\}}\mathbf{1}_{\{\tau^B(j)\leq t,\forall j\in\mathcal{D}\}}\right]\\
&= \mathbb{E}^0\left[E_t^{\mathcal{S}}\exp\left(-\sum_{k\in\mathcal{C}}\int_0^t \alpha_s(k)A_s^{\mathcal{S}}(k)ds\right)\mathbf{1}_{\{\tau(k)>t,\forall k\in\mathcal{C}\}}\mathbf{1}_{\{\tau^B(j)\leq t,\forall j\in\mathcal{D}\}}\right]\\
&= \mathbb{E}^0\left[E_t^{\mathcal{S}}\exp\left(-\sum_{k\in\mathcal{C}}\int_0^t \alpha_s(k)A_s^{\mathcal{S}}(k)ds\right)\prod_{k\in\mathcal{C}}Z_t(k)\mathbf{1}_{\{\tau^B(j)\leq t,\forall j\in\mathcal{D}\}}\right].
\end{aligned}$$

The last equality is obtained by using the fact that $e(i), i \in \mathcal{N}$ are independent under $\mathbb{P}^0$ and $\mathcal{C}\cap\mathcal{S} = \emptyset$:

$$\begin{aligned}
\mathbb{P}^0(\tau(k) > t, \forall k \in \mathcal{C}|\mathcal{G}_t^{\mathcal{S}}) &= \mathbb{E}^0\left[\mathbb{P}^0\left(\tau(k) > t, \forall k \in \mathcal{C}|\mathcal{F}_t \vee_{i\in\mathcal{S}} \sigma(e(i))\right)|\mathcal{G}_t^{\mathcal{S}}\right]\\
&= \mathbb{E}^0\left[\mathbb{P}^0\left(\tau(k) > t, \forall k \in \mathcal{C}|\mathcal{F}_t\right)|\mathcal{G}_t^{\mathcal{S}}\right] = \prod_{k\in\mathcal{C}}\mathbb{P}^0\left(\tau(k) > t|\mathcal{F}_t\right)
\end{aligned}$$

To continue, we just need to use the expression for $Z(k)$ in (3.2) and for $\bar{\mathbb{P}}_{\mathcal{C}}$ in Definition 3.4:

$$\begin{aligned}
p^{\mathcal{S}|\mathcal{D}}(t) &= \mathbb{E}^0\left[E_t^S \exp\left(-\sum_{k\in\mathcal{C}}\int_0^t \left[\lambda_s(k)+\alpha_s(k)A_s^{\mathcal{S}}(k)\right]ds\right)\prod_{k\in\mathcal{C}}\mathcal{E}_t(\nu(k))\mathbf{1}_{\{\tau^B(j)\le,\forall j\in\mathcal{D}\}}\right]\\
&= \bar{\mathbb{E}}_{\mathcal{C}}\left[E_t^S \exp\left(-\sum_{k\in\mathcal{C}}\int_0^t \left[\lambda_s(k)+\alpha_s(k)A_s^{\mathcal{S}}(k)\right]ds\right)\mathbf{1}_{\{\tau^B(j)\le t,\forall j\in\mathcal{D}\}}\right]\\
&= \bar{\mathbb{E}}_{\mathcal{C}}\left[F_t^{\mathcal{S}} \exp\left(-\sum_{k\in\mathcal{C}}\Lambda_t^{\mathcal{S}}(k)\right)\mathbf{1}_{\{\tau^B(j)\le t,\forall j\in\mathcal{D}\}}\right]\\
&= \bar{\mathbb{E}}^{\mathcal{C}}\left[L_t^{\mathcal{S}}\mathbf{1}_{\{\tau^B(j)\le t,\forall j\in\mathcal{D}\}}\right],
\end{aligned}$$

so that (5.8) is proved. We now prove the particular formulas of our proposition:

(a) The formula (5.2) is obtained from (5.8) with $\mathcal{D}=\emptyset$.

(b) On the set $\{\tau^B(j)<t;\forall j\in\mathcal{D}\}$ we have that $L^{\mathcal{S}}$, which is a $\mathbb{G}^{\mathcal{S}}$-adapted process, is equal to some $\mathbb{G}^{\mathcal{S}-\mathcal{D}}$-adapted process, that is:

$$L_t^{\mathcal{S}}\mathbf{1}_{\{\tau^B(j)<t;\forall j\in\mathcal{D}\}} = L_t^{\mathcal{S}|\mathcal{D}}\mathbf{1}_{\{\tau^B(j)<t;\forall j\in\mathcal{D}\}}, \tag{5.9}$$

with $L^{\mathcal{S}|\mathcal{D}}$ being $\mathbb{G}^{\mathcal{S}-\mathcal{D}}$-adapted. We proceed to identify the process $L^{\mathcal{S}|\mathcal{D}}$ (basically this consists in replacing $\tau(j)$ with $T(j)$, as they are equal on the set $\{\tau^B(j)<\infty\}$). We need to show it corresponds to the expression in (5.4).

First, we notice that:

$$\begin{aligned}
&\prod_{i\in\mathcal{S}}\mathcal{E}_t\left(\int_0^{\cdot} A_s^{\mathcal{S}}(i)dm_s(i)\right)\mathbf{1}_{\{\tau^B(j)<t;\forall j\in\mathcal{D}\}} =\\
&= \exp\left(-\sum_{j\in\mathcal{D}}\int_0^{t\wedge T(j)} A_s^{\mathcal{S}-\mathcal{D}}(j)\alpha(j)ds\right)\prod_{i\in\mathcal{S}-\mathcal{D}}\mathcal{E}_t\left(\int_0^{\cdot} A_s^{\mathcal{S}-\mathcal{D}}(i)dm_s(i)\right)\mathbf{1}_{\{\tau^B(j)<t;\forall j\in\mathcal{D}\}}
\end{aligned}$$

Also, for $k\in\mathcal{S}-\mathcal{D}$, we have:

$$\begin{aligned}
B_t^{\mathcal{S}}(k)\mathbf{1}_{\{\tau^B(j)<t;\forall j\in\mathcal{D}\}}(k) &= B_t^{\mathcal{S}|\mathcal{D}}(k)\mathbf{1}_{\{\tau^B(j)<t;\forall j\in\mathcal{D}\}}\\
\lambda_t^{S}(k)\mathbf{1}_{\{\tau^B(j)<t;\forall j\in\mathcal{D}\}} &= \lambda_t^{\mathcal{S}|\mathcal{D}}(k)\mathbf{1}_{\{\tau^B(j)<t;\forall j\in\mathcal{D}\}}.
\end{aligned}$$

with $B^{\mathcal{S}|\mathcal{D}}$ and $\lambda^{\mathcal{S}|\mathcal{D}}$ being $\mathbb{G}^{\mathcal{S}-\mathcal{D}}$-adapted; $B^{\mathcal{S}|\mathcal{D}}$ is given in (5.7), and $\lambda^{\mathcal{S}|\mathcal{D}}$ in (5.6). We therefore identify $L^{\mathcal{S}|\mathcal{D}}$ as the one in (5.4).

Using the relation (5.9) in the formula (5.8) and then the fact that $L^{\mathcal{S}|\mathcal{D}}$ is $\mathbb{G}^{\mathcal{S}-\mathcal{D}}$ adapted, we obtain:

$$\begin{aligned}
p^{\mathcal{S}|\mathcal{D}}(t) &= \bar{\mathbb{E}}_{\mathcal{C}}\left[L_t^{\mathcal{S}|\mathcal{D}}\mathbf{1}_{\{\tau^B(j)\le t,\forall j\in\mathcal{D}\}}\right]\\
&= \bar{\mathbb{E}}_{\mathcal{C}}\left[L_t^{\mathcal{S}|\mathcal{D}}\bar{\mathbb{P}}_{\mathcal{C}}\left(\tau^B(j)\le t,\forall j\in\mathcal{D}|\mathcal{G}_t^{\mathcal{S}-\mathcal{D}}\right)\right]\\
&= \bar{\mathbb{E}}_{\mathcal{C}}\left[L_t^{\mathcal{S}|\mathcal{D}}\bar{\mathbb{P}}_{\mathcal{C}}\left(\tau^B(j)\le t,\forall j\in\mathcal{D}|\mathcal{F}_t\right)\right]\\
&= \bar{\mathbb{E}}_{\mathcal{C}}\left[\ell_t^{\mathcal{S}|\mathcal{D}}\bar{\mathbb{P}}_{\mathcal{C}}\left(\tau^B(j)\le t,\forall j\in\mathcal{D}|\mathcal{F}_t\right)\right].
\end{aligned}$$

On the other hand,

$$\bar{\mathbb{P}}_{\mathcal{C}}\left(\tau^B(j) \le t, \forall j \in \mathcal{D} | \mathcal{F}_t\right) = \mathbb{P}^0\left(\tau^B(j) \le t, \forall j \in \mathcal{D} | \mathcal{F}_t\right)$$
$$= \prod_{j \in \mathcal{D}} \mathbb{P}^0\left(\tau^B(j) \le t | \mathcal{F}_t\right) = \prod_{j \in \mathcal{D}} p_t(j) \mathbf{1}_{\{T(j) \le t\}},$$

so that (5.8) is proved.

□

The dynamics of the processes $\ell^{\mathcal{S}}$ and $\ell^{\mathcal{S}|\mathcal{D}}$ will be obtained from intermediary quantities, falling basically into two categories:

Proposition 5.4. *The following hold, for $j \in \mathcal{S} - \mathcal{D}$:*

$$(a)\ \bar{\mathbb{E}}_{\mathcal{C}}[L_t^{\mathcal{S}|\mathcal{D}} \mathbf{1}_{\{\tau^B(j)<t\}} | \mathcal{F}_t] = \ell_t^{\mathcal{S}|\mathcal{D}\cup j} p_t(j) \mathbf{1}_{\{T(j)<t\}}, \tag{5.10}$$
$$(b)\ \bar{\mathbb{E}}_{\mathcal{C}}[L_t^{\mathcal{S}|\mathcal{D}} \mathbf{1}_{\{\tau^A(j)<t\}} | \mathcal{F}_t] = \ell_t^{\mathcal{S}|\mathcal{D}} - \ell_t^{\mathcal{S}-j|\mathcal{D}} - \ell_t^{\mathcal{S}|\mathcal{D}\cup j} p_t(j) \mathbf{1}_{\{T(j)<t\}}. \tag{5.11}$$

We recall that $\ell^{\mathcal{S}|\mathcal{D}}$ is the $(\mathbb{F}, \bar{\mathbb{P}}_{\mathcal{C}})$-optional projection of the process $L^{\mathcal{S}|\mathcal{D}}$. Consequently, $\ell^{\mathcal{S}-j|\mathcal{D}}$ is the $(\mathbb{F}, \bar{\mathbb{P}}_{\mathcal{C}\cup\{j\}})$-optional projection of the process $L^{\mathcal{S}-j|\mathcal{D}}$.

Proof. We prove (5.10).

$$\bar{\mathbb{E}}_{\mathcal{C}}[L_t^{\mathcal{S}|\mathcal{D}} \mathbf{1}_{\{\tau^B(j)<t\}} | \mathcal{F}_t] = \bar{\mathbb{E}}_{\mathcal{C}}[L_t^{\mathcal{S}|\mathcal{D}\cup j} \mathbf{1}_{\{\tau^B(j)<t\}} | \mathcal{F}_t]$$
$$= \bar{\mathbb{E}}_{\mathcal{C}}\left[L_t^{\mathcal{S}|\mathcal{D}\cup j} \bar{\mathbb{P}}_C\left(\tau^B(j) < t | \mathcal{G}^{\mathcal{S}-\mathcal{D}-j}\right) | \mathcal{F}_t\right]$$
$$= \bar{\mathbb{E}}_{\mathcal{C}}\left[L_t^{\mathcal{S}|\mathcal{D}\cup j} p_t(j) \mathbf{1}_{\{T(j)<t\}} | \mathcal{F}_t\right] = \bar{\mathbb{E}}_{\mathcal{C}}\left[L_t^{\mathcal{S}|\mathcal{D}\cup j} | \mathcal{F}_t\right] p_t(j) \mathbf{1}_{\{T(j)<t\}}$$
$$= \ell_t^{\mathcal{S}|\mathcal{D}\cup j} p_t(j) \mathbf{1}_{\{T(j)<t\}}.$$

On the other hand, we have

$$\{\tau^A(j) < t\} = \left(\{\tau(j) \ge t\} \cup \{\tau^B(j) < t\}\right)^c.$$

so that (again with $\mathcal{D} \subset \mathcal{S}$, $j \in \mathcal{S} - \mathcal{D}$):

$$\bar{\mathbb{E}}_{\mathcal{C}}[L_t^{\mathcal{S}|\mathcal{D}} \mathbf{1}_{\{\tau^A(j)<t\}} | \mathcal{F}_t] = \ell_t^{\mathcal{S}|\mathcal{D}} - \bar{\mathbb{E}}_{\mathcal{C}}[L_t^{\mathcal{S}|\mathcal{D}} \mathbf{1}_{\{\tau(j)\ge t\}} | \mathcal{F}_t] - \bar{\mathbb{E}}_{\mathcal{C}}[L_t^{\mathcal{S}|\mathcal{D}} \mathbf{1}_{\{\tau^B(j)<t\}} | \mathcal{F}_t]$$
$$= \ell_t^{\mathcal{S}|\mathcal{D}} - \ell_t^{\mathcal{S}-j|\mathcal{D}} - \ell_t^{\mathcal{S}|\mathcal{D}\cup j} p_t(j) \mathbf{1}_{\{T(j)<t\}}.$$

□

5.3. **Proof of Theorem 4.1.** The theorem is partially proved in Proposition 5.3, where also the processes $\ell^{\mathcal{S}|\mathcal{D}}$ and $\ell^{\mathcal{S}}$ are shown to be the $(\mathbb{F}, \bar{\mathbb{P}}_{\mathcal{C}})$ optional projections of $L^{\mathcal{S}|\mathcal{D}}$ and $L^{\mathcal{S}}$. It remains to prove the dynamics for $\ell^{\mathcal{S}|\mathcal{D}}$ and $\ell^{\mathcal{S}}$ are correct.

We notice that the stated dynamics for $\ell^{\mathcal{S}}$ coincides with the one of $\ell^{\mathcal{S}|\emptyset}$ derived from (4.4), when taking $\mathcal{D} = \emptyset$. Therefore, it is only needed to prove that for a general $\mathcal{D} \subset \mathcal{S}$ the dynamics for $\ell^{\mathcal{S}|\mathcal{D}}$ in (4.4) is correct.

To do so, we start from the SDE corresponding to $L^{\mathcal{S}|\mathcal{D}}$, which we then project on $\mathbb{F}$ under the measure $\bar{\mathbb{P}}_{\mathcal{C}}$. From (5.4), we have:

$$
\begin{aligned}
L_t^{\mathcal{S}|\mathcal{D}} = 1 &+ \sum_{i\in\mathcal{S}-\mathcal{D}} \int_0^t L_{s-}^{\mathcal{S}|\mathcal{D}} A_s^{\mathcal{S}-\mathcal{D}}(i) dm_s(i) + \sum_{i\in\mathcal{N}} \int_0^t L_{s-}^{\mathcal{S}|\mathcal{D}} B_s^{\mathcal{S}|\mathcal{D}}(i) d\bar{n}_s^{\mathcal{C}}(i) \\
&- \sum_{i\in\mathcal{C}} \int_0^t L_{s-}^{\mathcal{S}|\mathcal{D}} \lambda_s^{\mathcal{S}|\mathcal{D}}(i) ds - \sum_{i\in\mathcal{D}} \int_0^{t\wedge T(i)} L_{s-}^{\mathcal{S}|\mathcal{D}} \alpha_s(i) A_s^{\mathcal{S}-\mathcal{D}}(i) ds
\end{aligned}
$$

Therefore, for finding the process $\ell^{\mathcal{S}|\mathcal{D}}$, we compute the $(\mathbb{F}, \bar{\mathbb{P}}_{\mathcal{C}})$ optional projections of each term on the right hand side of the above expression. It is important to emphasise that the filtration $\mathbb{F}$ is immersed in the filtration $\mathbb{G}^{\mathcal{S}-\mathcal{D}}$ under the measure $\bar{\mathbb{P}}_{\mathcal{C}}$ (see Lemma 5.1 (c)). Therefore, we can use the classical projection formulas summarised in the Appendix (Proposition A.6 and Lemma A.7).

First, we have:

Lemma 5.5. *For all $i \in \mathcal{S} - \mathcal{D}$ and $t \geq 0$:*

$$
\bar{\mathbb{E}}_{\mathcal{C}} \left[\int_0^t L_{s-}^{\mathcal{S}|\mathcal{D}} A_s^{\mathcal{S}-\mathcal{D}}(i) dm_s(i) | \mathcal{F}_t \right] = 0.
$$

Proof. We fix some $i \in \mathcal{S}-\mathcal{D}$. We notice that $m(i)$ is a $(\mathbb{G}^{\mathcal{S}-\mathcal{D}}, \bar{\mathbb{P}}_{\mathcal{C}})$ martingale and the process $H(i) := L_{.-}^{\mathcal{S}|\mathcal{D}} A^{\mathcal{S}-\mathcal{D}}(i)$ is predictable. It follows that the optional projection of $\int H(i) dm(i)$ is null, as an application of the Lemma A.7. Indeed, taking $\rho = \tau^A(i)$, $\mathbb{H} := \mathbb{G}^{\mathcal{S}-\mathcal{D}-i}$, we observe that the conditions for applying Lemma A.7 are fulfilled: the filtrations $\mathbb{G}^{\mathcal{S}-\mathcal{D}-i}$ and $\mathbb{G}^{\mathcal{S}-\mathcal{D}}$ are immersed under $\bar{\mathbb{P}}_{\mathcal{C}}$, the process $H(i)$ is here bounded and $\tau^A(i)$ avoids all $\mathbb{G}^{\mathcal{S}-\mathcal{D}-i}$ stopping times (Corollary 3.3). □

Secondly:

Lemma 5.6. *For all $i \in \mathcal{N}$ and $t \geq 0$,*

$$
\bar{\mathbb{E}}_{\mathcal{C}} \left[\int_0^t L_{s^-}^{\mathcal{S}|\mathcal{D}} B_s^{\mathcal{S}|\mathcal{D}}(i) d\bar{n}_s^{\mathcal{C}}(i) | \mathcal{F}_t \right] = \int_0^t \bar{\mathbb{E}}_{\mathcal{C}} \left[L_{s^-}^{\mathcal{S}|\mathcal{D}} B_s^{\mathcal{S}|\mathcal{D}}(i) | \mathcal{F}_s \right] \, d\bar{n}_s^{\mathcal{C}}(i)
$$

Proof. It is a direct application of Proposition A.6, with $M := \bar{n}^{\mathcal{C}}(i)$ and $G = L_{s^-}^{\mathcal{S}|\mathcal{D}} B_s^{\mathcal{S}|\mathcal{D}}(i)$. □

It follows form the last two lemmas that the $\ell^{\mathcal{S}|\mathcal{D}}$ writes:

$$
\begin{aligned}
\ell_t^{\mathcal{S}|\mathcal{D}} = 1 &+ \sum_{i\in\mathcal{N}} \int_0^t \bar{\mathbb{E}}_{\mathcal{C}} \left[L_{s^-}^{\mathcal{S}|\mathcal{D}} B_s^{\mathcal{S}|\mathcal{D}}(i) | \mathcal{F}_s \right] \, d\bar{n}_s^{\mathcal{C}}(i) - \sum_{i\in\mathcal{C}} \int_0^t \bar{\mathbb{E}}_{\mathcal{C}} \left[L_{s^-}^{\mathcal{S}|\mathcal{D}} \lambda_s^{\mathcal{S}|\mathcal{D}}(i) | \mathcal{F}_s \right] ds \\
&- \sum_{i\in\mathcal{D}} \int_0^{t\wedge T(i)} \alpha_s(i) \bar{\mathbb{E}}_{\mathcal{C}} \left[L_{s^-}^{\mathcal{S}|\mathcal{D}} A_s^{\mathcal{S}-\mathcal{D}}(i) | \mathcal{F}_s \right] ds
\end{aligned}
\tag{5.12}
$$

The expression above contains some conditional expectations that we now compute explicitly, with the help of Proposition 5.4.

For $i \in \mathcal{D}$:

$$
\begin{aligned}
\bar{\mathbb{E}}_{\mathcal{C}}\left[L_{t^-}^{\mathcal{S}|\mathcal{D}}A_t^{\mathcal{S}-\mathcal{D}}(i)|\mathcal{F}_t\right] &= \bar{\mathbb{E}}_{\mathcal{C}}\left[L_{t-}^{\mathcal{S}|\mathcal{D}}\frac{1}{\alpha_t(i)}\sum_{j\in\mathcal{S}-\mathcal{D}}\phi_t^A(i,j)\mathbf{1}_{\{\tau^A(j)<t\}}|\mathcal{F}_t\right] \\
&= \frac{1}{\alpha_t(i)}\sum_{j\in\mathcal{S}-\mathcal{D}}\phi_t^A(i,j)\bar{\mathbb{E}}_{\mathcal{C}}\left[L_{t^-}^{\mathcal{S}|\mathcal{D}}\mathbf{1}_{\{\tau^A(j)<t\}}|\mathcal{F}_t\right] \\
&= \sum_{j\in\mathcal{S}-\mathcal{D}}\left(\ell_{t^-}^{\mathcal{S}|\mathcal{D}}-\ell_{t^-}^{\mathcal{S}-j|\mathcal{D}}-\ell_{t^-}^{\mathcal{S}|\mathcal{D}\cup j}p_t(j)\mathbf{1}_{\{T(j)<t\}}\right)\frac{\phi_t^A(i,j)}{\alpha_t(i)}
\end{aligned}
$$

On the other hand, for $i\in\mathcal{N}$:

$$
\begin{aligned}
\bar{\mathbb{E}}_{\mathcal{C}}\left[L_{t^-}^{\mathcal{S}|\mathcal{D}}B_t^{\mathcal{S}|\mathcal{D}}(i)|\mathcal{F}_t\right] &= \frac{1}{\gamma_t(i)}\bar{\mathbb{E}}_{\mathcal{C}}\left[L_{t^-}^{\mathcal{S}|\mathcal{D}}\left(\sum_{j\in\mathcal{S}-\mathcal{D}}\phi_t^B(i,j)\mathbf{1}_{\{\tau^B(j)<t\}}+\sum_{j\in\mathcal{D}}\phi_t^B(i,j)\mathbf{1}_{\{T(j)<t\}}\right)|\mathcal{F}_t\right] \\
&= \sum_{j\in\mathcal{S}}\left(\mathbf{1}_{\{j\in\mathcal{D}\}}\ell_{t^-}^{\mathcal{S}|\mathcal{D}}+\mathbf{1}_{\{j\in\mathcal{S}-\mathcal{D}\}}\ell_{t^-}^{\mathcal{S}|D\cup j}p_t(j)\right)\frac{\phi_t^B(i,j)}{\gamma_t(i)}\mathbf{1}_{\{T(j)<t\}}.
\end{aligned}
$$

Finally, for $i\in\mathcal{C}$, and using the two above computed quantities:

$$
\begin{aligned}
&\bar{\mathbb{E}}_{\mathcal{C}}\left[L_{t^-}^{\mathcal{S}|\mathcal{D}}\lambda_t^{\mathcal{S}|\mathcal{D}}(i)|\mathcal{F}_t\right] = \\
&= \bar{\mathbb{E}}_{\mathcal{C}}\left[L_{t^-}^{\mathcal{S}|\mathcal{D}}\left\{\lambda_t(i)+\alpha_t(i)A_t^{\mathcal{S}-\mathcal{D}}(i)+\beta_t(i)B_t^{\mathcal{S}|\mathcal{D}}(i)\right\}|\mathcal{F}_t\right] \\
&= \ell_{t^-}^{\mathcal{S}|\mathcal{D}}\lambda_t(i)+\alpha_t(i)\bar{\mathbb{E}}_{\mathcal{C}}\left[L_{t^-}^{\mathcal{S}|\mathcal{D}}A_t^{\mathcal{S}-\mathcal{D}}(i)|\mathcal{F}_t\right]+\beta_t(i)\bar{\mathbb{E}}_{\mathcal{C}}\left[L_{t^-}^{\mathcal{S}|\mathcal{D}}B_t^{\mathcal{S}-\mathcal{D}}(i)|\mathcal{F}_t\right] \\
&= \ell_{s^-}^{\mathcal{S}|\mathcal{D}}\left(\lambda_t(i)+\phi_s^A(i,\mathcal{S}-\mathcal{D})+g_t(i)\sum_{j\in\mathcal{D}}\phi^B(i,j)\mathbf{1}_{\{T(j)<t\}}\right) \\
&\quad -\sum_{j\in\mathcal{S}-\mathcal{D}}\left[\ell_{t^-}^{S-j|\mathcal{D}}\phi_t^A(i,j)+\ell_{t^-}^{\mathcal{S}|\mathcal{D}\cup j}\left[\phi_t^A(i,j)-g_t(i)\phi_t^B(i,j)\right]p_t(j)\mathbf{1}_{\{T(j)<t\}}\right].
\end{aligned}
$$

We now replace the conditional expectations in (5.12) with the terms computed above; we obtain the following:

$$
\begin{aligned}
d\ell_t^{\mathcal{S}|\mathcal{D}} &= \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{S}}\left(\mathbf{1}_{\{j\in\mathcal{D}\}}\ell_{t^-}^{\mathcal{S}|\mathcal{D}}+\mathbf{1}_{\{j\in\mathcal{S}-\mathcal{D}\}}\ell_{t^-}^{\mathcal{S}|\mathcal{D}\cup j}p_t(j)\right)\frac{\phi_t^B(i,j)}{\gamma_t(i)}\mathbf{1}_{\{T(j)<t\}}d\bar{n}_t^{\mathcal{C}}(i) \\
&-\sum_{i\in\mathcal{C}}\left\{\ell_{t^-}^{\mathcal{S}|\mathcal{D}}\left(\lambda_t(i)+\phi_s^A(i,\mathcal{S}-\mathcal{D})+g_t(i)\sum_{j\in\mathcal{D}}\phi^B(i,j)\mathbf{1}_{\{T(j)<t\}}\right)\right. \\
&\qquad\left.-\sum_{j\in\mathcal{S}-\mathcal{D}}\left[\ell_{s^-}^{S-j|\mathcal{D}}\phi_s^A(i,j)+\ell_{s^-}^{\mathcal{S}|\mathcal{D}\cup j}\left[\phi_s^A(i,j)-g_s(i)\phi_s^B(i,j)\right]p_s(j)\mathbf{1}_{\{T(j)<s\}}\right]\right\}dt \\
&-\sum_{i\in\mathcal{D}}\mathbf{1}_{\{T(i)>t\}}\sum_{j\in\mathcal{S}-\mathcal{D}}\phi_t^A(i,j)\left(\ell_{t-}^{\mathcal{S}|\mathcal{D}}-\ell_t^{\mathcal{S}-j|\mathcal{D}}-\ell_t^{\mathcal{S}|\mathcal{D}\cup j}p_t(j)\mathbf{1}_{\{T(j)<t\}}\right)dt
\end{aligned}
$$

We rearrange terms and use:

$$\bar{n}^{\mathcal{C}}(i) = \begin{cases} n(i) + \int_0^{t\wedge T(i)} g_s(i)ds & i \in \mathcal{C} \\ n(i) & i \in \mathcal{S} \end{cases}$$

which follows from (5.1) and the remark thereafter. Also, we denote

$$\psi^A(i,j) := \begin{cases} \phi^A(i,j) & i \in \mathcal{S} - \mathcal{D} \\ \phi^A(i,j)\mathbf{1}_{\{T(i)>t\}} & i \in \mathcal{D} \end{cases}$$

We obtain the dynamics of $\ell^{\mathcal{S}|\mathcal{D}}$:

$$\begin{aligned} d\ell_t^{\mathcal{S}|\mathcal{D}} &= \ell_{t-}^{\mathcal{S}|\mathcal{D}} \left\{ \left[\lambda_t(\mathcal{C}) + \psi_t^A(\mathcal{C}\cup\mathcal{D}, \mathcal{S}-\mathcal{D})\right] dt + \sum_{j\in\mathcal{D}} \mathbf{1}_{\{T(j)<t\}} \sum_{i\in\mathcal{N}} \left(\frac{\phi_t^B(i,j)}{\gamma_t(i)}\right) dn_t(i) \right\} \\ &\quad + \sum_{j\in\mathcal{S}-\mathcal{D}} \ell_t^{\mathcal{S}|\mathcal{D}\cup j} p_t(j) \mathbf{1}_{\{T(j)<t\}} \left\{ \psi^A(\mathcal{C}\cup\mathcal{D}, j)dt + \sum_{i\in\mathcal{N}} \left(\frac{\phi_t^B(i,j)}{\gamma_t(i)}\right) dn_t(i) \right\} \\ &\quad + \sum_{j\in\mathcal{S}-\mathcal{D}} \ell_t^{\mathcal{S}-j|\mathcal{D}} \psi^A(\mathcal{C}\cup\mathcal{D}, j)dt. \end{aligned}$$

This is nothing but another form of (4.4), so that the result is proved.

Appendix A. Basic facts in enlargement of filtrations

This Appendix summarises the results from the theory of enlargements of a filtration that were useful in this paper.

We assume we are given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{H} = (\mathcal{H}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual assumptions.

A. **Progressive enlargement.**

Definition A.1. A random time ρ is a nonnegative random variable $\rho : (\Omega, \mathcal{F}) \to [0, \infty]$.

The Azéma supermartingale associated to ρ and relative to $(\mathbb{H}, \mathbb{P})$ is the $\mathbb{H}$ supermartingale

$$Z_t^\rho = \mathbf{P}(\rho > t \mid \mathcal{H}_t) \tag{A.1}$$

chosen to be càdlàg, associated with ρ by Azéma (Azéma [1]). We note that the supermartingale (Z_t^ρ) is the $\mathbb{H}$-optional projection of $\mathbf{1}_{[0,\rho[}$. We also introduce the $\mathbb{H}$ dual optional and dual predictable projections of the process $\mathbf{1}_{\{\rho\leq t\}}$, denoted respectively by A_t^ρ and a_t^ρ. Then,

$$Z_t^\rho = \mathbb{E}^{\mathbb{P}}[A_\infty^\rho | \mathcal{H}_t] - A_t^\rho.$$

while the Doob-Meyer decomposition of (A.1) writes:

$$Z_t^\rho = m_t^\rho - a_t^\rho. \tag{A.2}$$

We enlarge the initial filtration $\mathbb{H}$ with the process $(\rho \wedge t)_{t\geq 0}$, so that the new enlarged filtration $\mathbb{H}^\rho$ is the smallest filtration (satisfying the usual assumptions) containing $\mathbb{H}$ and making ρ a stopping time, that is:

$$\mathcal{H}_t^\rho = \mathcal{K}_{t+}, \text{ where } \mathcal{K}_t = \mathcal{H}_t \vee \sigma(\rho \wedge t).$$

Now we recall a theorem which is useful in constructing the $(\mathbb{H}^\rho, \mathbb{P})$ compensator process of ρ.

Theorem A.2 (Jeulin-Yor [30])**.** *Let H be a bounded $\mathbb{H}^\rho$ predictable process. Then*

$$H_\rho \mathbf{1}_{\{\rho \le t\}} - \int_0^{t\wedge\rho} \frac{H_s}{Z^\rho_{s-}} da^\rho_s$$

is a $\mathbb{H}^\rho$ martingale.

When one assumes that the random time ρ avoids $\mathbb{H}$ stopping times, then:

Lemma A.3 (Jeulin-Yor [30], Jeulin [29])**.** *If ρ avoids $\mathbb{H}$ stopping times, then $A^\rho = a^\rho$ and A^ρ is continuous. Therefore, the compensator of the process $\mathbf{1}_{\{\rho \le t\}}$ is continuous.*

B. **Immersion of filtrations.** Given two filtrations $\mathbb{H}$ and $\mathbb{G}$, with $\mathcal{H}_t \subset \mathcal{G}_t$, for all $t \ge 0$, the following assumption is often encountered in the literature:

The filtration $\mathbb{H}$ is immersed in $\mathbb{G}$ (also called (H)-hypothesis): every $\mathbb{H}$ martingale is a $\mathbb{G}$ martingale.

We write $\mathbb{H} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$ for $\mathbb{H}$ is immersed in $\mathbb{G}$ under the probability measure $\mathbb{P}$.

We now recall several useful equivalent characterizations of the immersion property in the next theorem

Theorem A.4 (Dellacherie-Meyer [13] and Brémaud-Yor [6])**.** *The following assertions are equivalent:*

(1) $\mathbb{H} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$ *;*

(2) For all bounded $\mathcal{H}_\infty$-measurable random variables H and all bounded $\mathcal{G}_t$-measurable random variables G_t, we have

$$\mathbb{E}^{\mathbb{P}}\left[HG_t|\mathcal{H}_t\right] = \mathbb{E}^{\mathbb{P}}\left[H|\mathcal{H}_t\right]\mathbb{E}^{\mathbb{P}}\left[G_t|\mathcal{H}_t\right].$$

(3) For all bounded $\mathcal{H}_\infty$ measurable random variables H,

$$\mathbb{E}^{\mathbb{P}}\left[H \mid \mathcal{G}_t\right] = \mathbb{E}^{\mathbb{P}}\left[H \mid \mathcal{H}_t\right].$$

The immersion property is preserved only by certain changes of the probability measure. One such example is the following:

Proposition A.5 (Jeulin-Yor [31])**.** *We assume that $\mathbb{H} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$. Let $\mathbb{Q}$ be a probability measure which is equivalent to $\mathbb{P}$ on $\mathcal{G}_\infty$. If $d\mathbb{Q}/d\mathbb{P}$ is $\mathcal{H}_\infty$-measurable, then $\mathbb{H} \overset{\mathbb{Q}}{\hookrightarrow} \mathbb{G}$.*

One advantage of the immersion property is that optional projections of some $\mathbb{G}$ adapted processes can be computed easily. We recall the projection formulas that were useful in the derivation of our main result.

Proposition A.6 (Brémaud-Yor [6])**.** *Suppose that $\mathbb{H} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$.*

(i) *Let M be an $\mathbb{H}$ local martingale and G be a $\mathbb{G}$ adapted and bounded process. Then the $\mathbb{H}$ optional projection of the process $\left(\int GdM\right)$ is given by $\int {}^oGdM$, where ${}^oG$ is the $\mathbb{H}$ optional projection of G.*

(ii) *If M is a $\mathbb{G}$ square integrable martingale and H an $\mathbb{H}$ adapted and bounded process. Then the $\mathbb{H}$ optional projection of the process $\left(\int HdM\right)$ is given by $\int Hd\,{}^oM$, where ${}^oM$ is the $\mathbb{H}$ optional projection of M.*

In the framework and with the notations of the previous subsection, we have:

Lemma A.7 (Coculescu et al. [8]). *Assume that ρ avoids all $\mathbb{H}$ stopping times and $\mathbb{H} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{H}^{\rho}$ holds. Let H be a $\mathbb{G}$-predictable process and let $N_t = \mathbf{1}_{\{\rho \le t\}} - \Gamma_{t\wedge\rho}$ be a $\mathbb{G}$ martingale. If $\mathbb{E}^{\mathbb{P}}[|H_\rho|] < \infty$, then the $\mathbb{H}$ optional projection of the process $\left(\int HdN\right)$ is null.*

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